THE COLLEGES OF OXFORD UNIVERSITY

MATHEMATICS

SUNDAY 10 DECEMBER, 2000

Time allowed: $2\frac{1}{2}$ hours

For candidates applying for Mathematics, Computer Science, Mathematics & Computer Science, or Mathematics & Philosophy

Write your name, college (where you are sitting the test), and proposed course (from the list above) in BLOCK CAPITALS.

NAME:

COLLEGE:

COURSE:

Attempt all the questions. Each of the ten parts of Question 1 is worth 4 marks, and Questions 2,3,4,5 are worth 15 marks each, giving a total of 100.

Question 1 is a multiple choice question for which marks will be given solely for the correct answers. Answer Question 1 on the grid on Page 2. Write your answers to Questions 2,3,4,5 in the space provided, continuing on the back of this booklet if necessary.

THE USE OF CALCULATORS OR FORMULA SHEETS IS PROHIBITED

1. For each part of the question on Pages 3 - 7, you will be given four possible answers, just one of which is correct. Indicate for each part A – J which answer (a), (b), (c), or (d) you think is correct with a tick (√) in the corresponding column in the table below. You may use the spaces between the parts for any rough working.

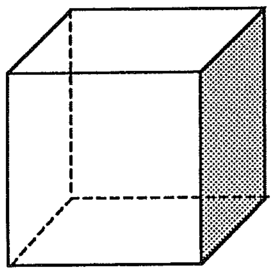
	(a)	(b)	(c)	(d)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

A. The substitution $x = y + t$ transforms the equation $x^3 + ax^2 + bx + c = 0$ into an equation of the form $y^3 + py + q = 0$ when

(a) $t = \frac{a}{3}$ (b) $t = -\frac{a}{3}$ (c) $t = a$ (d) $t = -a$.

B. The faces of a cube are coloured red or blue. Exactly three are red and three are blue. The number of distinguishable cubes that can be produced (allowing the cube to be turned around) is

(a) 2 (b) 4 (c) 6 (d) 20.



C. The shortest distance from the origin to the line $3x+4y=25$ is

(a) 3 (b) 4 (c) 5 (d) 6.

D. The numbers 10, 11 and –12 are solutions of the cubic equation

(a) $x^3-11x^2-122x+1320=0$

(b) $x^3-9x^2+122x-1320=0$

(c) $x^3-9x^2-142x+1320=0$

(d) $x^3+9x^2-58x-1320=0$.

E. The maximum gradient of the curve $y = x^4 - 4x^3 + 4x^2 + 2$ in the range $0 \le x \le 2\frac{1}{5}$ occurs when

(a) $x = 0$ (b) $x = 1 - \frac{1}{\sqrt{3}}$ (c) $x = 1 + \frac{1}{\sqrt{3}}$ (d) $x = 2\frac{1}{5}$.

F. The expression $x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2 - x^3 - y^3 - z^3 - 2xyz$ factorises as

(a) $(x+y+z)(x-y+z)(-x+y-z)$

(b) $(x+y-z)(x-y-z)(-x+y+z)$

(c) $(x+y-z)(x-y+z)(-x+y+z)$

(d) $(x-y-z)(-x-y+z)(-x+y-z)$.

G. The derivative of $xe^{-x^2}\cos\left(\frac{1}{x}\right)$ is

(a) $-\frac{1}{x}e^{-x^2}\sin\left(\frac{1}{x}\right)-2x^2e^{-x^2}\cos\left(\frac{1}{x}\right)+e^{-x^2}\cos\left(\frac{1}{x}\right)$

(b) $\frac{1}{x}e^{-x^2}\sin\left(\frac{1}{x}\right)-2x^2e^{-x^2}\cos\left(\frac{1}{x}\right)+e^{-x^2}\cos\left(\frac{1}{x}\right)$

(c) $\frac{1}{x}e^{-x^2}\sin\left(\frac{1}{x}\right)+2x^2e^{-x^2}\cos\left(\frac{1}{x}\right)+e^{-x^2}\cos\left(\frac{1}{x}\right)$

(d) $\frac{1}{x}e^{-x^2}\cos\left(\frac{1}{x}\right)-2x^2e^{-x^2}\cos\left(\frac{1}{x}\right)+e^{-x^2}\cos\left(\frac{1}{x}\right)$.

H. You are *told* that the infinite series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots \quad \text{and} \quad 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$$

have sums $\frac{\pi^2}{6}$ and $\frac{\pi^2}{8}$ respectively. The infinite series

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots+(-1)^{n-1}\frac{1}{n^2}+\cdots$$

has sum equal to

(a) $\frac{\pi^2}{9}$ (b) $\frac{\pi^2}{10}$ (c) $\frac{\pi^2}{12}$ (d) $\frac{\pi^2}{16}$.

I. A grid of size 3 cm $\times$ 5 cm is drawn, ruled at 1 cm intervals. The number of squares that can be drawn using the grid lines is

(a) 15 (b) 18 (c) 26 (d) 37.

J. A pack of cards consists of 52 different cards. A malicious dealer changes one of the cards for a second copy of another card in the pack and he then deals the cards to four players, giving thirteen to each. The probability that one players has two identical cards is

(a) $\frac{3}{13}$ (b) $\frac{12}{51}$ (c) $\frac{1}{4}$ (d) $\frac{13}{51}$.

2. Let p and q be real numbers. Show that the graph

$$y = x^3 + px + q$$

has turning points if and only if p is negative.

Assume now that p is negative. Find the values of y at the turning points and hence show that the equation

$$x^3 + px + q = 0$$

has three (distinct) real roots if and only if $27q^2 < -4p^3$.

[You may find it helpful to sketch the graph, clearly indicating the turning points.]

3. (a) Find the coordinates of the points at which the two curves $y = 6x^2$ and $y = x^4 - 16$ intersect.

(b) Give a *rough* sketch of the two curves (in the same diagram) for the range $-3 \le x \le 3$.

(c) Find the area of the region enclosed by the two curves.

4. (a) It is known that differentiation satisfies the four rules

(1) $$\frac{d}{dx}(\text{constant}) = 0,$$

(2) $$\frac{d}{dx}(x) = 1,$$

(3) $$\frac{d}{dx}(af(x) + bg(x)) = a\frac{df}{dx} + b\frac{dg}{dx}$$ for any constants a, b, and

(4) $$\frac{d}{dx}(f(x).g(x)) = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}.$$

.From these rules alone show that, for $n = 1,2,3,\ldots,$

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Use Rule (4) to find the derivative of the function

$$f(x) = \frac{1}{x^n}.$$

(It will help you to notice that $x^n.\frac{1}{x^n} = 1$.)

(b) A careless calculus student remembers Rules (1), (2) and (3) correctly, but thinks that Rule (4) says

(4′) $$\frac{d}{dx}(f(x).g(x)) = \frac{df}{dx} + f(x)\frac{dg}{dx}.$$

What will he or she compute for

$$\frac{d}{dx}\left(x^4\right)$$

5. A set of 12 rods, each 1 metre long, is arranged so that the rods form the edges of a cube. Two corners, A and B, are picked with AB the diagonal of a face of the cube.

An ant starts at A and walks along the rods from one corner to the next, never changing direction while on any rod. The ant's goal is to reach the corner B. A *path* is any route taken by the ant in travelling from A to B.

(a) What is the length of the shortest path, and how many such shortest paths are there?

(b) What are the possible lengths of paths, starting at A and finishing at B, for which the ant does not visit any vertex more than once (including A and B)?

(c) How many different possible paths of greatest length are there in (b)?

(d) Can the ant travel from A to B by passing through every other vertex exactly *twice* before arriving at B without revisiting A? Give brief reasons for your answer.